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The development of Poiseuille flow

By S. WILSON

Department of Mathematics, University of Manchester

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The problem considered is that of two-dimensional viscous flow in a straight channel. The steady Navier–Stokes equations are linearized on the assumption of small disturbance from the fully developed flow, leading to an eigenvalue equation resembling the Orr–Sommerfeld equation. This is solved in the limiting cases of small and large Reynolds number R, and an approximate method is proposed for moderate R. The main results are (i) the dominant mode of the disturbance velocity (i.e. that which persists longest) is antisymmetrical; (ii) for large R there are two sequences of eigenvalues. Both sequences are asymptotically real as $R \to \infty$. The members of the first sequence are $O(1)$ as $R \to \infty$ and are complex for all finite R. The members of the second sequence are $O(R^{-1})$ and the imaginary part is $O(R^{-N})$ for all N. It is the eigenvalues of the second sequence which will dominate the flow at large R.

1. Introduction

The problem of flow in the inlet region in pipes and channels is one of obvious practical interest and has received much attention in the past. The early work on entry flow in a circular pipe is summarized in Rosenhead (1963); more detail is given in Goldstein (1965) and the somewhat simpler problem of two-dimensional flow in a straight channel is also considered. In all this work the Reynolds number is assumed to be large. The basis of the method is to divide the entry region into zones; near the entrance the flow is assumed to consist of an inviscid core together with a thin boundary layer on the walls; far from the entrance, the solution is obtained as a perturbation of the fully developed flow. The two solutions are then patched together at some intermediate location.

A different approach has been used by Sparrow, Lin & Lundgren (1964). Here the equations are first simplified on the assumption of large Reynolds number and then made linear by introducing a suitably stretched downstream co-ordinate. The stretching factor is determined from the requirement that the local pressure gradient calculated from momentum considerations be the same as that calculated from energy considerations. This paper also contains a comprehensive survey of the experimental work.

In the present work attention is confined to the two-dimensional problem of flow in a straight channel; an attempt is made to solve the Navier–Stokes equations in the region some distance from the inlet, where the flow has almost attained the fully developed velocity distribution. The equations are linearized on the assumption of small disturbance from the parallel flow, leading to an eigenvalue equation for the decay of a stationary perturbation very similar to the Orr–Sommerfeld equation.

In § 3 this equation is solved in the limiting case of small Reynolds number. The next two sections deal with the case of large Reynolds number. The analysis of § 4 is based on the similarity of the present equation to the Orr–Sommerfeld equation and the aim has been to adapt the considerable body of analysis relating to the stability problem to the present purpose.

A numerical attempt on this problem has been made by Gillis & Brandt (1964). This work will be discussed in the conclusion (§ 7); the results given there indicate that there is a sequence of eigenvalues which for large R are distinct from those found in § 4 and the appropriate analysis is given in § 5.

2. The perturbation equation

The channel width is taken to be $2h$, the flux per unit span Q, and the kinematic viscosity ν. Then with length scale h, velocity scale Q/h and Reynolds number $R = Q/\nu$ the dimensionless streamfunction ψ satisfies the equation

$$\left(\psi_y \frac{\partial}{\partial x} - \psi_x \frac{\partial}{\partial y}\right) \nabla^2\psi = \frac{1}{R}\nabla^4\psi, \tag{2.1}$$

where x is the (dimensionless) downstream co-ordinate and y is the (dimensionless) transverse co-ordinate. With the origin in the centre of the channel the boundary conditions on the walls $y = \pm 1$ are

$$\psi(x, \pm 1) = \pm 1, \quad \psi_y(x, \pm 1) = 0.$$

The velocity far downstream approaches the parabolic profile and so

$$\left.\begin{aligned} \psi \to \psi_\infty(y) &= \tfrac{1}{2}(3y - y^3) \\ \psi_x &\to 0 \end{aligned}\right\} \quad \text{as} \quad x \to \infty.$$

The problem is completely posed if ψ and ψ_x are prescribed as functions of y at some value of x which may be taken as $x = 0$. It is obvious that if

$$\psi(0, y) = \tfrac{1}{2}(3y - y^3), \quad \psi_x(0, y) = 0$$

then the solution is simply $\psi = \frac{1}{2}(3y - y^3)$; the assumption now is that, if the boundary conditions are slightly perturbed, the solution is also slightly perturbed. More precisely, if

$$\psi(0, y) = \tfrac{1}{2}(3y - y^3) + \epsilon f(y),$$

$$\psi_x(0, y) = \epsilon g(y),$$

then
$$\psi = \tfrac{1}{2}(3y - y^3) + \epsilon\tilde{\psi}(x, y), \tag{2.2}$$

for small ϵ. Substituting (2.2) in (2.1) and neglecting squares of ϵ leads to a linear equation for $\tilde{\psi}$ whose coefficients are independent of x; writing

$$\tilde{\psi} = \phi(y) \exp(-\alpha x)$$

leads finally to

$$\phi^{\text{iv}} + 2\alpha^2\phi'' + \alpha^4\phi + \alpha R\{\tfrac{3}{2}(1-y^2)(\phi''+\alpha^2\phi)+3\phi\} = 0, \tag{2.3}$$

with boundary conditions $\phi(\pm 1) = \phi'(\pm 1) = 0.$

The similarity to the Orr–Sommerfeld equation (Rosenhead 1963, p. 515) will be noted. The main difference is that in the present equation α is the eigenvalue, not a prescribed wave-number; the equation is non-linear in α, which in general will be complex. It is readily shown that, if α is an eigenvalue, so is α^* (the complex conjugate); the eigenfunction is ϕ^*. A second difference is that we are here interested only in decaying modes, since growing modes cannot satisfy the boundary condition at $x = \infty$; this simplification is possible because of the elliptic character of the equations (cf. the unsteady Orr–Sommerfeld equation which is parabolic in t). Since there is symmetry about the real axis, as just noted, attention may be confined to the first quadrant of the α plane. There is in fact an infinite sequence of eigenvalues here for each fixed R, which may be ordered by the magnitude of the real part. The corresponding eigenfunctions will be alternately odd or even functions of y (since each one has more zero in $-1 < y < 1$ than the last) and the terms odd and even will be applied to the eigenvalues themselves.

The objective, then, is to calculate α for all R; this can be done in the extreme cases $R \to \infty$ and $R \ll 1$, and for intermediate R some approximate method must be used. The corresponding eigenfunctions are of less interest; the equation is not self-adjoint, that they do not form an orthonormal set, which means that there is no expansion theorem for the prescribed functions on the boundary $x = 0$. Physically, the most interesting problem is to find the eigenvalue with smallest real part, since this component of the disturbance persists longest.

3. Theory for small R

Since the coefficients in (2.3) are entire functions of y, α and R the solution will have the same properties and the following expansions may be used:

$$\left.\begin{aligned} \phi &= \phi_0 + R\phi_1 + \ldots,\\ \alpha &= \alpha_0 + R\alpha_1 + \ldots. \end{aligned}\right\} \tag{3.1}$$

Substituting in (2.3) and retaining only zero-order terms gives

$$\left.\begin{aligned} \phi_0^{\text{iv}} + 2\alpha_0^2\phi_0'' + \alpha_0^4\phi_0 &= 0,\\ \phi_0(\pm 1) = \phi_0'(\pm 1) &= 0. \end{aligned}\right\} \tag{3.2}$$

This problem has an analogue in the theory of plane elastostatics and this has received considerable attention (e.g. Johnson & Little 1964), mainly with a view to finding expansions in terms of the eigenfunctions of (3.2). This is of secondary interest in the present problem and the analysis needed is sufficiently straightforward to be presented here. The eigenvalue α_0 must be a root of

$$\sin 2\alpha_0 = \pm 2\alpha_0, \tag{3.3}$$

with the positive sign giving the odd eigenvalues. All the roots are complex except $\alpha_0 = 0$, which may be ignored because the corresponding ϕ_0 is identically zero. The roots of (3.3) have been tabulated (Hillman & Salzer 1943, Robbins & Smith 1948); if α_0 is a root then so are α_0^* and $-\alpha_0$, giving a root in each quadrant of the α plane.

It is an interesting fact that the first-quadrant eigenvalue with smallest real part is an *even* eigenvalue; this means that the corresponding streamwise velocity perturbation is *odd*. In the literature attention has been concentrated on symmetrical velocity perturbations but it now appears that the first odd perturbation has the smallest decay rate. This may be understood if it is recalled that the perturbations are made at constant flux; it follows that any symmetric streamwise velocity perturbation must have at least two zeros in $-1 < y < 1$; the first asymmetrical mode has only one.

Proceeding to the next order of approximation, we find

$$\phi_1^{\text{iv}} + 2\alpha_0^2\phi_1'' + \alpha_0^4\phi_1 = -\tfrac{3}{2}\alpha_0(1-y^2)(\phi_0''+\alpha_0^2\phi_0) - 3\alpha_0\phi_0 - 4\alpha_0\alpha_1(\phi_0''+\alpha_0^2\phi_0), \quad (3.4)$$

$$\phi_1(\pm 1) = \phi_1'(\pm 1) = 0.$$

The right-hand side is a known function of y except for α_1. The fact that α_0 is an eigenvalue of the homogeneous equation leads to a condition to be imposed on the right-hand side in order for a solution to exist. This may be obtained by multiplying by ϕ_0 and integrating from -1 to $+1$, or merely by solving the equation by elementary methods and applying the boundary conditions. For the even eigenvalues it turns out to be

$$F_e(1)(1-2i\alpha_0-a^2) = 2F_e'(1),$$

and for the odd ones

$$F_o(1)(1-2i\alpha_0+a^2) = 2F_o'(1),$$

where $a = \exp(i\alpha_0)$ and

$$F_e(y) = \left\{3i\alpha_0(1+a^2)(\tfrac{1}{2}y^2-\tfrac{1}{12}y^4)+\tfrac{3}{2}(1-a^2)y^2+4i\alpha_0\alpha_1(1+a^2)y^2 + \frac{15}{4}\frac{i}{\alpha_0}(1+a^2)y^2\right\}\cos\alpha_0 y + \tfrac{3}{2}(1+a^2)y^3\sin\alpha_0 y,$$

$$F_o(y) = \left\{3i\alpha_0(1-a^2)(\tfrac{1}{2}y^2-\tfrac{1}{12}y^4)+\tfrac{3}{2}(1+a^2)y^2+4i\alpha_0\alpha_1(1-a^2)y^2 + \frac{15}{4}\frac{i}{\alpha_0}(1-a^2)y^2\right\} i\sin\alpha_0 y + \tfrac{3}{2}(1-a^2)y^3\cos\alpha_0 y.$$

In this way α_1 may be obtained in terms of α_0; the results for the first few eigenvalues are given in table 1.

Eigenvalue	α_0	α_1
1	$2\cdot10620+1\cdot12537i$	$-0\cdot34138+0\cdot13667i$
2	$3\cdot74884+1\cdot38434i$	$-0\cdot29461+0\cdot03815i$
3	$5\cdot35627+1\cdot55158i$	$-0\cdot27548+0\cdot01611i$
4	$6\cdot94998+1\cdot67611i$	$-0\cdot26633+0\cdot00836i$
5	$8\cdot53668+1\cdot77554i$	$-0\cdot26131+0\cdot00492i$
6	$10\cdot11926+1\cdot85838i$	$-0\cdot25828+0\cdot00315i$

TABLE 1. Eigenvalues for small R; $\alpha = \alpha_0 + R\alpha_1$

4. Theory for large R; eigenvalues of $O(1)$

In this section an asymptotic theory for large R is developed on the assumption that $\alpha = O(1)$ as $R \to \infty$. The analysis is modelled on Lin (1955, chapter 8) and is carried out using the techniques of matched asymptotic expansions as described by Van Dyke (1964). It is convenient therefore to take α^2 and αR as the parameters of the problem in order to keep the analysis as similar as possible to that used in the stability problem. The equation connecting α^2 and αR can then be solved for α in terms of R.

Two complementary asymptotic theories are needed. In the central region of the channel there are four solutions, asymptotically valid for fixed y as $\alpha R \to \infty$. These are suitable except near the walls, where there are 'critical layers'; in these regions four regular solutions are obtained, valid for fixed $s(\alpha R)^{\frac{1}{3}}$ as $\alpha R \to \infty$, where s is the distance from the wall.

In view of the symmetry of the problem, it is necessary to consider only half the channel, say $-1 \leqslant y \leqslant 0$. The four boundary conditions (two each at $y = 0$ and $y = -1$) together with the matching conditions enable the eigenvalue to be calculated.

We write $\epsilon = (\alpha R)^{-\frac{1}{3}}$; α is of course complex; it may be assumed that

$$0 \leqslant \arg \alpha \leqslant \tfrac{1}{2}\pi$$

as noted in § 1, and so the relevant branch of ϵ is $0 \leqslant \arg \epsilon \leqslant \frac{1}{6}\pi$.

The first pair of solutions is obtained from the expansions

$$\left.\begin{aligned} \phi &= \phi^{(0)} + \epsilon \ln \epsilon \phi^{(1)} + \epsilon \phi^{(2)} + \ldots, \\ \alpha^2 &= \alpha_0^2 + \epsilon \ln \epsilon \alpha_1^2 + \epsilon \alpha_2^2 + \ldots. \end{aligned}\right\} \tag{4.1}$$

The logarithmic term might not be anticipated at this stage but turns out to be necessary for the matching process. Writing $\mathscr{D}$ for the differential operator

$$\frac{d^2}{dy^2} + \alpha_0^2 + \frac{2}{1-y^2},$$

we find

$$\mathscr{D}\phi^{(0)} = 0, \tag{4.2}$$

$$\mathscr{D}\phi^{(1)} = -\alpha_1^2 \phi^{(0)}, \tag{4.3}$$

$$\mathscr{D}\phi^{(2)} = -\alpha_2^2 \phi^{(0)}. \tag{4.4}$$

It will appear that the main problem is to determine the boundary conditions on these equations. For the moment we note that $y = 0$ is an ordinary point of (4.2) and two independent solutions in powers of y can be obtained:

$$F_1(y) = 1 - \tfrac{1}{2}(\alpha_0^2 + 2)\, y^2 + \tfrac{1}{24}\alpha_0^2(\alpha_0^2 + 4)\, y^4 + \ldots, \tag{4.5}$$

$$F_2(y) = y - \tfrac{1}{6}(\alpha_0^2 + 2)\, y^3 + \tfrac{1}{120}(\alpha_0^4 + 4\alpha_0^2 - 8)\, y^5 + \ldots. \tag{4.6}$$

The second pair of solutions derived by Lin (1955), the viscous solutions, will not be needed. The leading terms behave like $\exp(\pm iQ(y)\epsilon^{-\frac{3}{2}})$, where Q is a real function of y; i.e. both solutions would be exponentially large (because ϵ is complex) and we merely require that they occur in that combination which is exponentially small, and can therefore be neglected in comparison with all the terms of (4.1).

The first approximation to the general solution in the outer region may now be written

$$\phi^{(0)} = AF_1(y) + BF_2(y),$$

and the boundary conditions are $\phi = \phi'' = 0$ at $y = 0$ for the odd eigenvalues and $\phi' = \phi''' = 0$ at $y = 0$ for the even ones, giving $A = 0$ and $B = 0$ respectively. Similar conditions must be satisfied by the higher-order terms. Thus one boundary condition on (4.2), (4.3) and (4.4) has been obtained, namely that $\phi^{(0)}$, $\phi^{(1)}$ and $\phi^{(2)}$, or their first derivatives, vanish at $y = 0$, according as the eigenvalue is odd or even.

The other boundary condition will come from matching with the inner solution. For this it will be necessary to obtain series solutions of (4.2), (4.3) and (4.4) about the singular point $y = -1$. Two independent solutions of (4.2) in terms of $(= y+1)$ are

$$f_1(s) = s - \tfrac{1}{2}s^2 - \tfrac{1}{6}\alpha_0^2 s^3 + \dots, \tag{4.7}$$

$$f_2(s) = 1 + \dots - f_1(s)\ln s. \tag{4.8}$$

These series will also give the complementary functions of (4.3) and (4.4); the general solution of the equation $\mathscr{D}u = g(s)$ (where $\mathscr{D}$ is rewritten in terms of s) is in fact

$$u(s) = Af_1(s) + Bf_2(s) + f_1(s)\int_0^s f_2(t)\,g(t)\,dt - f_2(s)\int_0^s f_1(t)\,g(t)\,dt. \tag{4.9}$$

This formula may be used to obtain expansions of $\phi^{(1)}$ and $\phi^{(2)}$ near $s = 0$; the details will be given later.

We turn now to the analysis of the appropriate 'inner' expansion. A stretched co-ordinate $\eta = s/\epsilon$ is introduced and solutions in powers of ϵ are sought:

$$\phi = \epsilon(g^{(0)}(\eta) + \epsilon g^{(1)}(\eta) + \dots). \tag{4.10}$$

The functions $g^{(0)}$ and $g^{(1)}$ satisfy the equations

$$Dg^{(0)} = 0, \tag{4.11}$$

$$Dg^{(1)} = \left(\tfrac{3}{2}\eta^2 \frac{d^2}{d\eta^2} + 3\right) g^{(0)}, \tag{4.12}$$

where D is the differential operator $(d^4/d\eta^4) + 3\eta(d^2/d\eta^2)$. At $\eta = 0$ we have $\phi = \phi' = 0$ and two independent solutions of (4.10) are

$$\left.\begin{aligned} g_1^{(0)} &= \int_0^\eta d\eta \int_0^\eta \mathrm{Ai}\,(-3^{\frac{1}{3}}\eta)\,dy, \\ g_2^{(0)} &= \int_0^\eta d\eta \int_0^\eta \mathrm{Bi}\,(-3^{\frac{1}{3}}\eta)\,d\eta, \end{aligned}\right\} \tag{4.13}$$

where Ai and Bi are the Airy functions. It is customary to write these solutions in terms of Hankel functions of order $\frac{1}{3}$, but there seems to be no particular advantage in this, and there is sufficient literature on the Airy functions for the present purpose (Miller 1946, Jeffreys & Jeffreys 1962).

We now require the asymptotic forms of $g_1^{(0)}$ and $g_2^{(0)}$ as $\eta \to \infty$; the relevant asymptotic expansions of the Airy functions are

$$\mathrm{Ai}(-\zeta) = \pi^{-\frac{1}{2}}\zeta^{-\frac{1}{4}}\sin\left(\tfrac{2}{3}\zeta^{\frac{3}{2}}+\tfrac{1}{4}\pi\right)\left(1+O(\zeta^{-\frac{3}{2}})\right),$$
$$\mathrm{Bi}(-\zeta) = \pi^{-\frac{1}{2}}\zeta^{-\frac{1}{4}}\cos\left(\tfrac{2}{3}\zeta^{\frac{3}{2}}+\tfrac{1}{4}\pi\right)\left(1+O(\zeta^{-\frac{3}{2}})\right),$$

which are uniformly valid for $-\frac{2}{3}\pi < \arg\zeta < \frac{2}{3}\pi$.

However, both functions are exponentially large as $\zeta \to \infty$ except when $\arg\zeta = 0$ (which implies that ϵ is real). In this case both functions oscillate increasingly rapidly as $\zeta \to \infty$, and matching with the outer solution would be impossible. The outer solution was derived on the assumption that ϵ has a non-zero imaginary part; otherwise it too would oscillate rapidly as $\epsilon \to 0$. This circumstance may be provisionally ruled out on physical grounds, and this assumption will be verified *a posteriori*. It is necessary, then, that the functions $g_1^{(0)}$ and $g_2^{(0)}$ be combined in such a way that the exponentially large term is absent; writing

$$\phi = \epsilon(a g_1^{(0)} + b g_2^{(0)}) + \ldots,$$

we see that the condition is $a = ib$. Since any normalization of the equations is acceptable we may take $b = 1$ and perform the integrations in (4.13). Including the terms from $g^{(1)}$ we find, as $\eta \to \infty$,

$$\phi \sim 2\times 3^{-\frac{4}{3}}i(\epsilon\eta - \tfrac{1}{2}\epsilon^2\eta^2) + 3^{-\frac{2}{3}}\epsilon(\mathrm{Bi}'(0)+i\mathrm{Ai}'(0))(1-\epsilon\eta\ln\eta)$$
$$+\text{exponentially small terms in } \eta + O(\epsilon^3).$$

Use has been made of the results

$$\int_0^\infty \mathrm{Ai}(-\theta)\,d\theta = \tfrac{2}{3}, \quad \int_0^\infty \mathrm{Bi}(-\theta)\,d\theta = 0 \quad (\theta\,\text{real}).$$

This series is to be matched with the outer solution given by (4.1) as $s \to 0$. It will be convenient to anticipate the results of the matching process somewhat. Using (4.9) with

$$g(s) = k_1 f_1(s) + k_2 f_2(s)$$

for some k_1, k_2, it is easily shown that the particular integrals of (4.3) and (4.4) are $O(s^2)$ as $s \to 0$. These terms will therefore not be needed for the first step of the matching process. This will show that $\phi^{(0)}$ contains no multiple of $f_2(s)$ (i.e. $k_2 = 0$) and so the particular integrals are in fact $O(s^3)$ as $s \to 0$. The outer solution may therefore be obtained to the required order without the particular integrals. We have

$$\phi = (A_0 + \epsilon\log\epsilon A_1 + \epsilon A_2 + \ldots)(s - \tfrac{1}{2}s^2 + \ldots)$$
$$+ (B_0 + \epsilon\log\epsilon B_1 + \epsilon B_2 + \ldots)(1 + \ldots - s\log s + \ldots). \quad (4.14)$$

The first step of the matching process (Van Dyke 1964, p. 93) gives

$$B_0 = 0, \quad A_0 = 2\times 3^{-\frac{4}{3}}i.$$

The boundary conditions on $\phi^{(0)}$ are therefore

$$\phi^{(0)} = 0 \quad \text{at} \quad s = 0, \phi^{(0)} \qquad \text{or} \qquad d\phi^{(0)}/ds = 0 \quad \text{at} \quad s = 1.$$

The eigenvalue α_0 may now be calculated numerically in a straightforward manner. The next stage of the matching process gives

$$B_1 = 0, \quad A_1 = 3^{-\frac{2}{3}}(\mathrm{Bi}'(0) + i\mathrm{Ai}'(0)).$$

The function $\phi^{(1)}$ therefore satisfies the same boundary conditions as $\phi^{(0)}$, and since α_0 is an eigenvalue of the homogeneous equation we see from (4.9) that a solution of (4.3) can exist only if

$$\alpha_1^2 \int_0^1 \{f_1(s)\}^2 ds = 0,$$

which implies $\alpha_1^2 = 0$ and $\phi^{(1)}$ is a multiple of $\phi^{(0)}$.

Finally we obtain from the matching that

$$B_2 = 3^{-\frac{2}{3}}(\mathrm{Bi}'(0) + i\mathrm{Ai}'(0)).$$

The function $\phi^{(2)}$ therefore satisfies

$$\phi^{(2)} = B_2 \quad \text{at} \quad s = 0, \qquad \phi^{(2)} = 0 \qquad \text{or} \qquad d\phi^{(2)}/ds = 0 \quad \text{at} \quad s = 1.$$

Using (4.9) and the fact that $\phi^{(0)} = A_0 f_1(s)$ we find

$$\alpha_2^2 \int_0^1 \{f_1(s)\}^2 ds = \tfrac{1}{2} \times 3^{\frac{2}{3}}(i\mathrm{Bi}'(0) - \mathrm{Ai}'(0)).$$

The integral here is readily found numerically and this gives α_2^2 in terms of α_0.

It now remains to solve the equation

$$\alpha^2 = \alpha_0^2 + \epsilon\alpha_2^2$$

for α in terms of R. It will be noticed that $\alpha_0 = 0$ is an eigenvalue of (4.2) and this case is slightly different from the higher eigenvalues. The result is

$$\left.\begin{aligned} \alpha &= (\alpha_2^2)^{\frac{6}{7}} R^{-\frac{1}{7}} + \ldots && (\alpha_0 = 0), \\ \alpha &= \alpha_0 + \tfrac{1}{2}\alpha_2^2 \alpha_0^{-\frac{4}{3}} R^{-\frac{1}{3}} + \ldots && (\alpha_0 \neq 0). \end{aligned}\right\} \tag{4.15}$$

The numerical details are given in table 2.

α_0	$a_2^2 \exp(-\frac{1}{3}i\pi)$	K
2·589	14·63	2·058
4·319	30·42	2·162
5·971	51·77	2·389
7·590	78·05	2.617
12·377	189·3	3·306

TABLE 2. Eigenvalues of $O(1)$ as $R \to \infty$. $\alpha^2 = \alpha_0^2 + \epsilon\alpha_2^2$; for $\alpha_0 \neq 0$, $\alpha = \alpha_0 + K \exp(\frac{1}{3}i\pi) R^{-\frac{1}{3}}$; for $\alpha_0 = 0$, $\alpha = 3·309 \exp(\frac{2}{7}\pi i) R^{-\frac{1}{7}}$

5. Theory for large R: eigenvalues of order R^{-1}

The analysis presented here is motivated by the results of Gillis & Brandt (1964), whose work will be discussed in more detail in § 7. On the assumption that α is $O(R^{-1})$ as $R \to \infty$, the balance of terms in (2.3) is different from that found in § 4; using the expansions

$$\left.\begin{aligned} \phi &= \phi_0 + R^{-1}\phi_1 + R^{-2}\phi_2 + \ldots, \\ \alpha R &= \alpha_0 + R^{-1}\alpha_1 + R^{-2}\alpha_2 + \ldots, \end{aligned}\right\} \tag{5.1}$$

one finds in fact

$$L\phi_0 = 0, \tag{5.2}$$

$$L\phi_1 = -\alpha_1\{\tfrac{3}{2}(1-y^2)\,\phi_0'' + 3\phi_0\}, \tag{5.3}$$

and the boundary conditions are $\phi_n(\pm 1) = \phi_n'(\pm 1) = 0$ for $n = 0, 1, 2, \ldots$. Here L is the differential operator

$$\frac{d^4}{dy^4} + \alpha_0\left\{\tfrac{3}{2}(1-y^2)\frac{d^2}{dy^2} + 3\right\}.$$

It does not seem possible to obtain analytic solutions of (5.2) and numerical methods were used to find the eigenvalues α_0. It is possible, however, to show that all the even eigenvalues are real. The equation may be integrated once to give

$$\phi_0'''(y) + \alpha_0\{\tfrac{3}{2}(1-y^2)\,\phi_0'(y) + 3y\,\phi_0(y)\} = \text{const.} = \phi'''(1) = \phi'''(-1). \tag{5.4}$$

If ϕ_0 is even, every term on the left of (5.4) is odd and it follows that $\phi'''(1) = 0$. Multiplying (5.2) by $\phi_0^{*''}$ (where * denotes the complex conjugate), integrating from -1 to $+1$ and using the result just obtained leads to the conclusion that α_0 is real. The author has been unable to establish whether or not the odd eigenvalues are real, but all the eigenvalues found (by numerical methods) were in fact real. As in the previous sections, the first eigenvalue is even; the results are given in table 3.

Eigenvalue	α_0
1	14·45
2	18·81
3	48·87
4	57·52
5	104·43

TABLE 3. Eigenvalues of $O(R^{-1})$ as $R \to \infty$; $\alpha = \alpha_0 R^{-1} + O(R^{-3})$

The next approximation, α_1, may be found by multiplying (5.3) by $\tilde{\phi}_0$, the solution of the adjoint equation to (5.2), and integrating from -1 to $+1$; this shows in fact that $\alpha_1 = 0$, and ϕ_1 is a multiple of ϕ_0. An inspection of the higher approximations shows that all the odd α_n are zero and also that, if α_0 is real, so are all the subsequent even α_n. For example, ϕ_2 satisfies

$$L\phi_2 = -2\alpha_0^2\phi_0'' - \tfrac{9}{2}\alpha_0^3(1-y^2)\,\phi_0 - \alpha_2\{\tfrac{3}{2}(1-y^2)\,\phi_0'' + 3\phi_0\},$$

and, if α_0 is real, so are ϕ_0, $\tilde{\phi}_0$ and α_2.

6. Approximate solution

In the previous sections the eigenvalues have been calculated for extreme values of R, namely $R \ll 1$ and $R \to \infty$. It is plausible in view of the remarks of § 2 that, as R varies, α traces out a smooth curve in the complex plane (in fact, separate curves for each branch of the function $\alpha(R)$). The calculations of §§ 3–5 will give the end-points of these curves together with the initial gradients. In this section an approximate method is suggested, to give some idea of the behaviour of α for intermediate values of R.

The basis is simply to substitute a trial function $\phi(y)$ in (2.3) and integrate from $y = -1$ to $y = 1$, giving an equation connecting α and R. Two modifications are needed. First, if ϕ is an odd function all the terms disappear. Second, the results are rather poor (when compared with the exact results of § 3, for instance), probably because too much depends on guessing accurately the value of ϕ''' at the end-points. Accordingly the equation is first multiplied by a suitable function before the integration is carried out. For the even eigenvalues we use $(1-y^2)^2$, and for the odd eigenvalues we use $y(1-y^2)^2$. These functions vanish with their first derivatives at the end-points, thus reducing the importance of the behaviour of ϕ at these points, and also make every term in the equation even.

A suitable trial function for the first eigenvalue is $\phi = (1-y^2)^2$; for the second we may use $\phi = y(1-y^2)^2$. These satisfy the boundary conditions and have the correct number of zeros. On integrating from $y = -1$ to $y = 1$ the following equations are obtained:

$$22\alpha^4 - 132\alpha^2 + 693 + \alpha R(30\alpha^2 - 33) = 0, \tag{6.1}$$

$$26\alpha^4 - 572\alpha^2 + 6435 + \alpha R(30\alpha^2 - 273) = 0. \tag{6.2}$$

The first-quadrant roots of these equations can be found numerically; the results are given in table 4.

R	First eigenvalue	Second eigenvalue
0	$2{\cdot}0751 + 1{\cdot}1429i$	$3{\cdot}6560 + 1{\cdot}5382i$
0·5	$1{\cdot}9149 + 1{\cdot}1923i$	$3{\cdot}5167 + 1{\cdot}5544i$
1·0	$1{\cdot}7747 + 1{\cdot}2127i$	$3{\cdot}3871 + 1{\cdot}5564i$
1·5	$1{\cdot}6532 + 1{\cdot}2138i$	$3{\cdot}2672 + 1{\cdot}5471i$
2·0	$1{\cdot}5486 + 1{\cdot}2021i$	$3{\cdot}1565 + 1{\cdot}5285i$
2·5	$1{\cdot}4588 + 1{\cdot}1825i$	$3{\cdot}0546 + 1{\cdot}5026i$
5·0	$1{\cdot}1644 + 1{\cdot}0467i$	$2{\cdot}6591 + 1{\cdot}3083i$
10	$0{\cdot}9176 + 0{\cdot}8161i$	$2{\cdot}2321 + 0{\cdot}8189i$
20	$0{\cdot}7526 + 0{\cdot}5456i$	1·2376
50	0·5434	0·4738

TABLE 4. Approximate eigenvalues as functions of R

The values obtained here are presumably better for small R than for large; the result is obviously wrong as $R \to \infty$. They are also somewhat better for the first eigenvalue than for the second, the errors being about 3 % and 10 %. It appears that the error lies mainly in the argument of the eigenvalue.

One important source of error here has been the choice of a real trial function for ϕ; it is in fact complex. However, a calculation of the exact eigenfunction for $R = 0$ showed that the imaginary part was fairly small.

7. Conclusions

The eigenvalue problem for the decay rate of small stationary disturbances to Poiseuille flow has been solved in the cases $R \ll 1$ and $R \gg 1$. The principal results are as follows. (i) For small R, α is complex; if α is an eigenvalue so is α^*. (ii) For large R two sequences of eigenvalues are found. The first sequence consists of eigenvalues of $O(1)$ as $R \to \infty$ (an exception being the first member, which is zero); each member is asymptotically real but is complex for any finite R. All these eigenvalues except the first approach their asymptotic values α_0 in such a way that $\alpha - \alpha_0 \sim KR^{-\frac{1}{3}} \exp(\frac{1}{3}i\pi)$, where K is a real number.

The second sequence consists of eigenvalues which are $O(R^{-1})$ as $R \to \infty$. All the eigenvalues which have been calculated are real and it has been proved that all the even eigenvalues are real. In these cases every term in the asymptotic expansions of the eigenvalues is real; this does not of course prove that they are real for all sufficiently large R, only that the imaginary part is asymptotically smaller than any inverse power of R. The flow will be dominated by the members of this sequence for sufficiently large R and the disturbance will persist downstream a distance $O(R)$.

The presence of two sequences of eigenvalues is probably a consequence of the fact that (2.3) is non-linear in α. Possibly the situation is modelled by (6.1), where if one assumes $\alpha = O(1)$ as $R \to \infty$ one finds $\alpha^2 \to \frac{11}{10}$, but the assumption $\alpha = O(R^{-1})$ leads to $\alpha R \to 21$. (iii) The first eigenvalue (i.e. with smallest real part) is even, corresponding to an asymmetrical velocity perturbation. The explanation has been given in § 3; this possibility seems to have been overlooked in the literature. It has been shown that for large R the solution for this eigenvalue has two branches, both of which approach zero, one as $R^{-\frac{1}{7}}$ and the other as R^{-1}. As noted the behaviour of the flow will be dominated by the second branch and disturbances will persist for a distance of $O(R)$. Since the higher modes are more rapidly damped it also follows that any asymmetry in the velocity profile will become more pronounced.

These results may be compared with those of Gillis & Brandt (1964), who attempted, as part of their work, a direct numerical solution of (2.3). It should be noted that Gillis & Brandt are concerned only with the odd eigenfunctions since they assume a symmetrical velocity profile throughout. No comparison, therefore, is possible with the first (even) mode of the present work. The relevant results are contained in a table which is reproduced here for convenience (adapted to the present notation) as table 5. This gives the first odd eigenvalue as a function of R.

For small R the agreement with the present work is good. At $R = 0{\cdot}25$, for example, the discrepancy with the result in table 1 is less than $0{\cdot}1$ %. For large R, Gillis & Brandt discovered only the eigenvalue of the second sequence; the explanation is probably that the eigenfunctions of the first sequence have a

R	α	αR
0	$3{\cdot}7488+1{\cdot}3843i$	.
0·25	$3{\cdot}6767+1{\cdot}3914i$	.
2·5	$3{\cdot}1544+1{\cdot}2966i$	.
$R_{\text{crit}} = 8{\cdot}461$	2·6320	22·25
10	1·9101	19·10
25	0·7492	18·73
50	0·3758	18·79
100	0·1881	18·81
250	0·0753	18·81
Large R		18·81

TABLE 5. Second eigenvalue as a function of R; from Gillis & Brandt (1964), adapted and rounded off

boundary-layer character which a numerical technique would not pick up unless special measures were taken. However the results given agree well with table 3.

Gillis & Brandt also find that there is a critical Reynolds number above which the eigenvalue is real ($R_{\text{crit}} = 8{\cdot}46$ approximately). As noted earlier in this section, the present calculations accord with, but do not establish, this result. In this respect it is unfortunate that Gillis & Brandt do not give more detail in the crucial range $5 < R < 15$ say, or similar calculations for the other modes.

The behaviour of α as a function of R may be summarized as follows. For $R = 0$ there is an infinite sequence of complex eigenvalues (given in table 1) which may be ordered by the magnitude of the real part. As R increases, each eigenvalue traces out a curve in the α plane; at some value of R each curve divides into two branches. One branch of each curve approaches the origin as $R \to \infty$; if the conclusions of Gillis & Brandt are true of all the eigenvalues then this approach is along the real axis. The second branch of each curve approaches a point on the real axis; these points are the numbers α_0 in table 2. In general the curve approaches the point at an angle $\frac{1}{3}\pi$; the exception is the first eigenvalue $\alpha_0 = 0$, where the angle is $\frac{2}{7}\pi$.

The precise nature of the behaviour of α at moderate R is unknown. The value of R at which the curves divide is presumably less than $R_{\text{crit}} = 8{\cdot}46$ for the second eigenvalue and for the other modes there is no information at all. This range of R is beyond the reach of the analytical methods used here and will presumably require further numerical exploration.

One further feature of the results may be noted. This appears in the exact and approximate solutions for small R and in the work of Gillis & Brandt. As R increases from zero the imaginary part of α exhibits a small initial increase; this feature is present in all the cases for which the calculation has been performed, and appears to diminish in magnitude for the higher eigenvalues; see table 1. The physical significance of this phenomenon is not apparent to the author.

As explained in § 2 attention has been confined to those eigenvalues whose real part is positive and therefore give rise to exponentially decaying behaviour as $x \to \infty$. However, a referee has pointed out that the eigenvalues with negative real part may be used to treat a slightly different problem. We now consider a channel infinite in both directions with fully developed Poiseuille flow for all x. A small stationary disturbance is introduced at $x = 0$, for example, by means of a grid. The decay of the disturbance as $x \to \infty$ may be studied using the analysis presented above, but for the region $x < 0$ one needs the eigenvalues with negative real part. In this way a solution for the whole channel may be found in two parts, one part valid for $x > 0$ and the other for $x < 0$. Neither solution can be continued into the other across the line $x = 0$ because there is a singularity there corresponding to the introduction of momentum by the obstacle. (The original problem differs in that no introduction of momentum at $x = 0$ is contemplated; the solution for $x < 0$ would be obtained by analytic continuation of the solution for $x > 0$. The exponentially large behaviour indicates, as expected, that the linearization is not valid for x large and negative.)

To find the eigenvalues with negative real part we note first that the equations of §§ 3, 4 (corresponding to small R and the $O(1)$ eigenvalues for large R respectively) remain the same when the sign of x is reversed, to the first approximation. The leading terms of the required eigenvalues are thus obtained by reversing the signs of the results of §§ 3, 4.

The required eigenvalues of $O(R^{-1})$ will be the negative eigenvalues of (5.2), and it seems likely that there are none. It is possible to prove that no negative even eigenvalues exist. It has been shown in § 5 that, when ϕ_0 is even, α_0 is real; multiplying (5.2) by ϕ'' and integrating from $y = -1$ to $y = +1$ gives

$$-\int_{-1}^{+1} \phi'''^2\,dy + \tfrac{3}{2}\alpha_0 \left\{\int_{-1}^{+1} (1-y^2)\,\phi''^2\,dy - 2\int_{-1}^{+1} \phi'^2\,dy\right\} = 0,$$

the boundary terms making no contribution in this case. The coefficient of α_0 is always positive. To show this, put

$$I_1 = \int_{-1}^{+1} (1-y^2)\,\phi''^2\,dy, \quad I_2 = \int_{-1}^{+1} \phi'^2\,dy,$$

and $\lambda = I_1/I_2$. Then λ is stationary when $\delta\lambda = 0$, or

$$\delta I_1 - \lambda \delta I_2 = 0;$$

writing $\Psi = \phi'$, this reduces, after integration by parts, to

$$\frac{d}{dy}\left\{(y^2-1)\frac{d\Psi}{dy}\right\} + \lambda\Psi = 0, \quad \Psi(\pm 1) = 0. \tag{7.1}$$

The least value of λ is therefore the least eigenvalue of (7.1) (Legendre's equation), which is 2, and the required result follows.

Machine calculations have shown that there are no real eigenvalues in the range $-200 < \alpha_0 < 0$ and one may therefore reasonably conjecture that all the eigenvalues of (5.2) are real and positive.

It follows that when R is large the only eigenvalues whose real part is negative are those of $O(1)$. The disturbance therefore penetrates upstream in a distance of $O(1)$, in contrast with the downstream region, where the disturbance penetrates a distance $O(R)$.

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REFERENCES

GILLIS J. & BRANDT, A. 1964 Weizmann Institute, *AF EOAR* 63-73 SR-1.

GOLDSTEIN, S. 1965 *Modern Developments in Fluid Mechanics*, vol. I. New York: Dover.

HILLMAN, A. P. & SALZER, H. E. 1943 *Phil. Mag.* **34**, 575.

JEFFREYS, H. & JEFFREYS, B. S. 1962 *Mathematical Physics*. Cambridge University Press.

JOHNSON, M. W. & LITTLE, R. W. 1964 *M.R.C. Tech. Summary Reports, University of Wisconsin*, no. 478.

LIN, C. C. 1955 *Theory of Hydrodynamic Stability*. Cambridge University Press.

MILLER, J. C. P. 1946 *The Airy Integral. B.A. Mathematical Tables*, part-volume B.

ROBBINS, C. I. & SMITH, R. C. T. 1948 *Phil. Mag.* **39**, 1004.

ROSENHEAD, L. 1963 *Laminar Boundary Layers*. Oxford University Press.

SPARROW, E. M., LIN, S. H. & LUNDGREN, T. S. 1964 *Phys. Fluids* **7**, 338.

VAN DYKE, M. 1964 *Perturbation Methods in Fluid Mechanics*. New York: Academic Press.